Brain Surface Conformal Mapping and Brain Volumetric Harmonic Map with Variational Methods

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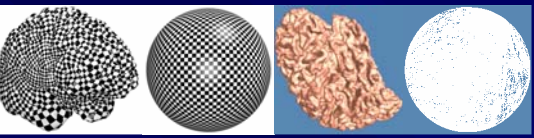
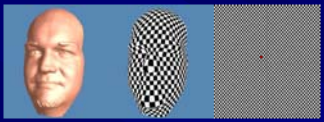


Abstract

We developed a general method for global surface conformal parameterization. For genus zero surface, we propose a new variational method which can find a unique conformal mapping between any two genus zero manifolds by minimizing the harmonic energy of the map. We apply this algorithm to the cortical surface matching problem. We further extended the algorithm to find a 3D volumetric harmonic map from a 3D brain volumetric model to a solid sphere. We use mesh structure to represent both surface and volumetric data. We proposed a new algorithm, sphere carving algorithm, which can build 3D tetrahedral mesh from a set of images with topology preservation. Experimental results on both synthetic data and real data are reported.

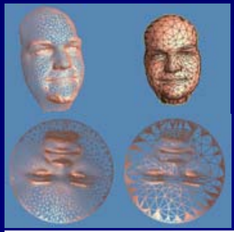
Conformal Mapping

- Any surface without holes or self-intersections can be mapped conformally onto the sphere
- This mapping, conformal equivalence, is one-to-one, onto, and angle preserving
- Locally, shape is preserved and distances and areas are only changed by a scaling factor
- A canonical space is useful for subsequent work



Genus Zero Surface Conformal Mapping Properties

- Intrinsic to geometry
- Independent of triangulation and resolution
- Depends on metric continuously
- Harmonic is equivalent to conformal
- All conformal are equivalent
- All the conformal construct a automorphism group: Möbius group which is a linear rational group on complex plane and a 6 dimensional group.



Harmonic Map

- The map minimizes the stretching energy
- Geodesics are harmonic maps from a circle to the surface
- In general, harmonic maps may not exist, or unique
- Depends on the Riemannian metrics. Independent of the embeddings
- For 3-manifold, the existence of harmonic maps, the uniqueness of harmonic maps, the diffeomorphic properties of harmonic maps are extremely difficult theoretic problems
- Between convex 3-disk, harmonic map exists and is most likely diffeomorphism

Surface Conformal Mapping Algorithm Details

- Minimize Harmonic Energy
- Use absolute derivative
- All computation are on the target surface, without projecting to complex plane
- Harmonic energy $f : M \to S^2$

$$E(f) = \int_M \|\nabla f\|^2 d\sigma_M$$

- Discrete harmonic energy

$$E(f) = \sum_{[u,v]\in M} k_{uv} \|f(u) - f(v)\|^2 \qquad k_{uv} = \tfrac{1}{2}(\cot\alpha + \cot\beta)$$

- Discrete Laplacian

$$\Delta f(u) = \sum_{[u,v]\in M} k_{uv}(f(u) - f(v))$$

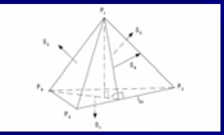
Volumetric Harmonic Map Algorithm Details

- Harmonic energy

$$E(f) = \langle f, f \rangle = \sum_{\{u,v\}\in K} k(u,v) \| f(u) - f(v) \|^2$$

$$k(u,v) = \frac{1}{12}\sum_{i=1}^{n} l_i \cot(\theta_i)$$

- Minimize Harmonic Energy



Sphere Carving Algorithm

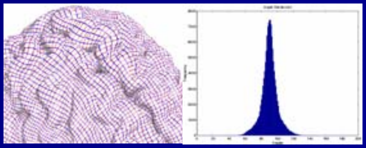
- Input (a sequence of volume images and a desired surface genus number)
- Output (a tetrahedral mesh whose surface has the desired genus number)

Build a solid handle body tetrahedral mesh consisted of tetrahedra, such that the sphere totally enclose the 3D data. Let the boundary of the solid sphere be S. We cut the model until we get the object 3D tetrahedral model.

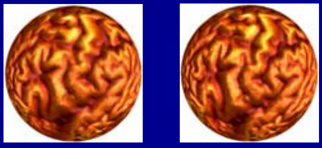
Experimental Results

(a) (b) (c) (d)

Reconstructed brain meshes and their spherical harmonic mappings. (a) and (c) are the reconstructed surfaces for the same brain scanned at different times. Due to scanner noise and inaccuracy in the reconstruction algorithm, there are visible geometric differences. (b) and (d) are the spherical conformal mappings of (a) and (c) respectively; the normal information is preserved. By the shading information, the correspondence is illustrated.



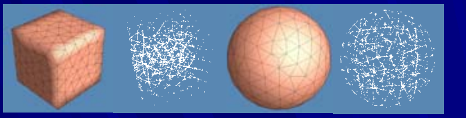
Conformality measurement. The curves of iso-polar angle and iso-azimuthal angle are mapped to the brain, and the intersection angles are measured on the brain. The histogram is illustrated.



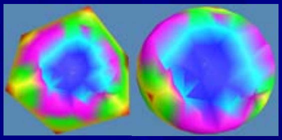
(a) (b)

Conformal mappings of surfaces with different resolutions. The original brain surface has 50,000 faces, and is conformally mapped to a sphere, as shown in (a). Then the brain surface is simplified to 20,000 faces, and its spherical conformal mapping is shown in (b).

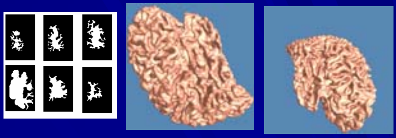
Volumetric Harmonic Map on Synthetic Data



Volumetric Parameterization

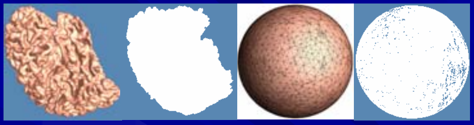


Sphere Carving Algorithm on Brain Model



Sphere Carving Algorithm on Prostate Model

Volumetric Brain Harmonic Map



More Genus Zero Surface Examples

More Volumetric Harmonic Map Examples